Causality, stability and passivity for a mirror in vacuum

Marc Thierry Jaekel [(a)] and Serge Reynaud [(b)]

(a) Laboratoire de Physique Théorique de l'ENS *, 24 rue Lhomond, F75231 Paris Cedex 05 France
(b) Laboratoire de Spectroscopie Hertzienne †, 4 place Jussieu, case 74, F75252 Paris Cedex 05 France

The mean force exerted upon a perfect mirror moving in vacuum in a two dimensional spacetime has the same expression as the radiation reaction force computed in classical electron theory. It follows that unacceptable runaway solutions are predicted. We show that this instability problem does not appear when partially transmitting mirrors are studied. The mechanical impedance describing the mirror coupled to vacuum radiation pressure is computed explicitly; recoil is neglected. It is found to be a passive function, so that stability is ensured. This is connected to the fact that no energy can be extracted from the vacuum state.

INTRODUCTION

A mirror moving in vacuum experiences a mean force which is associated through linear response theory with the quantum fluctuations of the vacuum stress tensor [1].

This motional force, denoted hereafter F_m, has been computed by Fulling and Davies [2] for a perfectly reflecting mirror scattering a scalar vacuum field in a two dimensional spacetime. A linear approximation in the mirror's displacement leads from their result to a force proportional to the third derivative of the time dependent mirror's position q

$$F_m(t) = \frac{\hbar q'''(t)}{6\pi c^2} \tag{1}$$

As required by Lorentz invariance of the vacuum state, this force vanishes for a uniform velocity. It also vanishes for a uniform acceleration so that the inertial mass of the mirror is not modified.

The motional force modifies the mechanical response of the mirror to an applied force F_a since the equation of motion can be written

$$kq(t) + mq''(t) = F_a(t) + F_m(t) \qquad k = m\omega_0^2 \tag{2}$$

where m is the mass of the mirror, ω_0 is the eigenfrequency of the suspension system ($\omega_0 = 0$ in the limiting case of a free mass) and F_m is the motional force.

Consequently, it has to be taken into account in a complete analysis of the sensitivity of interferometric measurements. This seems particularly important for interferometric detection of gravitational waves [3].

The force (1) has the same expression, with a modification of the numerical factor, as the radiation reaction force computed in classical electron theory [4]. It is well known that severe problems are associated with the resulting equation of motion (2): a historical account is given in ref. [4]; more recent references can be found in [5]. First, it turns out that 'runaway solutions' exist, which correspond to an exponential increase of the velocity in the absence of applied force. These runaway solutions may be prohibited by a special choice of boundary conditions, but a 'preacceleration' phenomenon then occurs, with an acceleration of the mass before the force is applied [4].

Clearly, the equation of motion (2) violates causality for a perfect mirror when the motional force is given by equation (1). We show in this letter that this problem is solved in a simple manner by considering partially transmitting mirrors rather than perfect ones. We will assume that the mirror is transparent at frequencies higher than a reflection cutoff ω_C and that this cutoff obeys the following inequality

$$\hbar\omega_C \ll mc^2 \tag{3}$$

This condition, obeyed by realistic mirrors (m is a macroscopic mass), implies that the recoil effect for reflected fields can be ignored in the derivation of the motional force [1].

Using a previously derived expression for the motional force associated with a partially transmitting mirror [1], we will demonstrate that the mirror coupled to the vacuum radiation pressure is a stable system (runaway solutions disappear). More precisely, we will show that the mirror obeys 'passivity' [6] when some inequality which is a consequence of condition (3) is satisfied. Passivity is closely connected to energy considerations: the moving mirror radiates energy into vacuum and is therefore

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[1]The perfect mirror corresponds to $\omega_C \to \infty$ and does not obey condition (3). It is therefore necessary to take the recoil into account, which was not done in the derivation of the force (1). This probably explains the inconsistencies associated with expression (1) of the force.

damped, while the vacuum cannot provide energy for exciting the mirror's motion. Then, stability follows and problems associated with runaway solutions are eliminated.

THE SUSCEPTIBILITY FUNCTION ASSOCIATED WITH THE MOTIONAL FORCE

At the limit of small displacements, the motional force is a function of q easily defined in the frequency domain in terms of a linear susceptibility χ

$$F_m(t) = \int dt' \chi(t - t') q(t') \qquad (4a)$$

$$F_m[\omega] = \chi[\omega] q[\omega] \qquad (4b)$$

Any function f is written in the time domain or in the frequency domain according to the rule

$$f(t) = \int \frac{d\omega}{2\pi} f[\omega] e^{-i\omega t}$$

The motional force (1) computed for a perfect mirror corresponds to the susceptibility

$$\chi[\omega] = \frac{i\hbar\omega^3}{6\pi c^2} = im\tau\omega^3 \qquad \tau = \frac{\hbar}{6\pi mc^2} \qquad (5)$$

where τ is a time constant, characteristic of the coupling of the mirror of mass m to the vacuum radiation pressure. The corresponding time constant for the electron is of the order of 10^{-22}s. It is much smaller for a macroscopic mirror.

In a scattering approach [7], a partially transmitting mirror is described by a reflectivity function $r[\omega]$ and a transmitivity function $s[\omega]$, which obey unitarity and causality conditions. The mirror is supposed to become transparent at the high frequency limit ($r \rightarrow 0$ when $\omega \rightarrow \infty$). The perfect mirror is then replaced by the more realistic model of a causal mirror having a perfect reflection below a reflection cutoff ω_C, but transparent at higher frequencies.

This approach has provided us with an expression for the mean Casimir force between two motionless mirrors [7], which is free from the divergences usually associated with the infiniteness of the vacuum energy, as well as a causal expression for the motional Casimir force between moving mirrors [8].

When condition (3) is obeyed, it is possible to ignore the recoil effect and the motional force can be written as [2]

$$\chi[\omega] = im\tau\omega^3 \Gamma[\omega] \qquad (6a)$$

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[2]The motional force for a mirror in vacuum is given by equations (33) and (11) of ref. [1]

where

$$\omega^3 \Gamma[\omega] = \int_0^\omega 3d\omega' \, (\omega - \omega')\omega' \alpha[\omega - \omega', \omega'] \qquad (6b)$$

$$\alpha[\omega, \omega'] = 1 - s[\omega]s[\omega'] + r[\omega]r[\omega'] \qquad (6c)$$

The function $\Gamma[\omega]$ is regular at the low frequency limit

$$\Gamma[\omega] \rightarrow \Gamma[0] = r[0]^2 \qquad \text{for } \omega \rightarrow 0$$

When $r[0] = -1$, the susceptibility (5) is recovered at low frequencies whatever the reflectivity may be at high frequencies. At high frequencies, $\Gamma[\omega]$ behaves as a cutoff function. Its behaviour will be discussed in more detail later on.

A simple example is provided by the following model fulfilling unitarity, causality and transparency conditions

$$r[\omega] = \frac{-1}{1 - \frac{i\omega}{\Omega}} \qquad s[\omega] = 1 + r[\omega] \qquad (7a)$$

One obtains in this case

$$\chi[\omega] = 6m\tau\Omega^3$$
$$\times \left(-\frac{i\omega}{\Omega} - \frac{\omega^2}{2\Omega^2} - \left(1 - \frac{i\omega}{\Omega}\right) \ln\left(1 - \frac{i\omega}{\Omega}\right) \right)$$

$$\Gamma[\omega] = 6\left(\frac{1}{2.3} + \frac{1}{3.4}\frac{i\omega}{\Omega} + \frac{1}{4.5}\left(\frac{i\omega}{\Omega}\right)^2 + \dots \right)$$
$$\text{for } \omega \ll \Omega \qquad (7b)$$

POSITIVITY OF THE DISSIPATIVE FUNCTIONS

Coming back to the general case, we introduce the real and imaginary parts of the functions written in the frequency domain, which are associated with dissipation and dispersion

$$\chi[\omega] = \chi_R[\omega] + i\chi_I[\omega] \qquad \Gamma[\omega] = \Gamma_R[\omega] + i\Gamma_I[\omega]$$
$$\chi_I[\omega] = m\tau\omega^3 \Gamma_R[\omega] \qquad \chi_R[\omega] = -m\tau\omega^3 \Gamma_I[\omega]$$

χ_R and Γ_R are even functions of ω while χ_I and Γ_I are odd functions of ω (see eqs 4, 6). The dissipative part χ_I of the susceptibility, which was denoted ξ_{FF} in ref. [1], is the commutator of the force operator.

Using the unitarity of the $S-$matrix associated with the mirror [1], one shows that

$$2\operatorname{Re}\alpha[\omega, \omega'] = |\alpha[\omega, \omega']|^2 + |\beta[\omega, \omega']|^2$$
$$\beta[\omega, \omega'] = s[\omega]r[\omega'] - r[\omega]s[\omega']$$

The following positivity property follows

$$\Gamma_R[\omega] \geq 0 \qquad \text{for } \omega \text{ real}$$
$$\chi_I[\omega] \geq 0 \qquad \text{for } \omega \text{ real} \geq 0 \qquad (8)$$

This property corresponds to the dissipative character (see for example ref. [9]) of the motional perturbation in the vacuum and will play an important role in the following.

DISPERSION RELATIONS

Causality [9] is associated with the fact that the susceptibility function, written in the time domain (see eq 4), vanishes for negative times

$$\chi(t) = 0 \qquad \text{for } t < 0$$

In the frequency domain, a causal function is analytic and regular in the half plane $\text{Im } \omega > 0$. For the susceptibility given by equations (6), these properties follow [1] from the causality of the scattering coefficients r and s.

The real and imaginary parts of a causal function are related through dispersion relations, which take the simple form of Kramers-Kronig relations when the function decreases sufficiently rapidly at high frequencies [9]. But this is not the case for the susceptibility function χ, so that it is necessary to write dispersion relations with subtractions [10].

From equations (6), one knows that the three quantities $\chi[0]$, $\chi'[0]$ and $\chi''[0]$ vanish. It is therefore tempting to write the dispersion relation with three subtractions at $\omega = 0$ which is just the Kramers-Kronig relation for the function Γ

$$\Gamma[\omega] = \int \frac{d\omega'}{i\pi} \frac{\Gamma_R[\omega']}{\omega' - \omega - i\epsilon} \qquad (9)$$

Assuming that $\Gamma[\omega]$ decreases sufficiently rapidly at high frequencies so that it is a square integrable function, the relation (9) is actually equivalent to causality for the susceptibility function. This follows from the Titchmarsh theorem [10].

As Γ_R is positive, one deduces that $\Gamma[\omega]$ behaves as $\frac{1}{\omega}$ at high frequencies

$$\Gamma[\omega] \approx \frac{\omega_C}{-i\omega} \qquad \text{for } \omega \to \infty \qquad (10)$$

where the parameter ω_C can be considered as the definition of the reflection cutoff (we suppose that the integral is finite)

$$\omega_C = \int \frac{d\omega}{\pi} \Gamma_R[\omega] = \int \frac{d\omega}{\pi} \frac{\chi_I[\omega]}{m\tau\omega^3} \qquad (11)$$

For example, the model of equations (7) corresponds to $\omega_C = 3\Omega$.

The behaviour of the susceptibility is described by a high frequency mass μ

$$\chi[\omega] \approx -\mu\omega^2 \qquad \text{for } \omega \to \infty$$
$$\mu = m\omega_C\tau = \frac{\hbar\omega_C}{6\pi c^2} \qquad (12)$$

The induced mass μ is quite similar to the 'electromagnetic mass' of electron theory [4]. It is positive (see eq 8) and much smaller than m for realistic macroscopic mirrors (see eq 3).

The dispersion relation (9) can be written equivalently in the time domain

$$\Gamma(t) = 2\theta(t)\Gamma_R(t)$$

It is thus clear that the susceptibility function does not obey the same simple dispersion relation; one can indeed write

$$\chi(t) = 2m\tau(\theta(t)\Gamma_R'''(t) + \delta(t)\Gamma_R''(0) \\ + \delta'(t)\Gamma_R'(0) + \delta''(t)\Gamma_R(0))$$

It follows that the expression (4a) of the motional force cannot be written simply in terms of the force commutator $\chi_I(t)$ since quasistatic motions must be subtracted up to the second order in a Mac-Laurin expansion. However, using the parity property of the function Γ_R, one recovers the consistency condition of linear response theory [3]

$$\chi(t) - \chi(-t) = 2m\tau\Gamma_R'''(t) = 2i\chi_I(t)$$

MECHANICAL IMPEDANCE OF THE MIRROR

When the coupling of the mirror to vacuum radiation pressure is taken into account, the equation of motion (2) is easily solved in the frequency domain.

The mechanical impedance Z gives the frequency components of the force F_a as a function of the velocity v

$$v(t) = q'(t) \qquad v[\omega] = -i\omega q[\omega]$$
$$F_a[\omega] = Z[\omega]v[\omega]$$
$$-i\omega Z[\omega] = k - m\omega^2 - \chi[\omega] \qquad (13)$$

The two real functions Z_R and Z_I are respectively the dissipative and dispersive parts of the impedance

$$Z[\omega] = Z_R[\omega] + iZ_I[\omega]$$
$$Z_R[\omega] = m\tau\omega^2\Gamma_R[\omega]$$
$$Z_I[\omega] = \frac{k}{\omega} - m\omega + m\tau\omega^2\Gamma_I[\omega] \qquad (14)$$

Equivalently, we can write the velocity v as a function of the applied force F_a and of the mechanical admittance Y

$$v[\omega] = Y[\omega]F_a[\omega] \qquad Y[\omega] = \frac{1}{Z[\omega]}$$

STABILITY

From causality, we know that the motional force is a retarded function of the mirror's position, which implies that the impedance function $Z[\omega]$ is analytic and has no poles in the half plane $\operatorname{Im}\omega > 0$.

The analytic properties of the admittance function $Y[\omega]$ require a closer examination since its expression in terms of the susceptibility $\chi[\omega]$ has the form of a 'closed loop gain', so that a self oscillation could occur. Such a self oscillation would be associated with the presence of a pole for Y, that is a zero for Z, in the half plane $\operatorname{Im}\omega > 0$.

But, if the suspended mirror coupled to the vacuum is a stable system, the self oscillation cannot take place. It follows that the admittance function $Y[\omega]$ shall also be analytic and regular in the half plane $\operatorname{Im}\omega > 0$ and that the impedance function $Z[\omega]$ should have no zero in this domain.

Using the susceptibility function (5) of a perfect mirror (with neglected recoil) and assuming that the mirror is free ($k = 0$), one checks that the function Y has a pole in the half plane $\operatorname{Im}\omega > 0$, at $\omega = \frac{i}{\tau}$, in this case. This is similar to the well known instability problem of electron theory [4].

This result is modified when a partially transmitting mirror is studied. Indeed, the pole associated with runaway solutions corresponds to a frequency well above the reflection cutoff (see eq 3). Due to the asymptotic behaviour of the cutoff function Γ (see eq 10), the pole which was located at $\omega = \frac{i}{\tau}$ for a perfect mirror is shifted to $\omega \approx -i\omega_C$ so that the instability is suppressed.

PASSIVITY

We demonstrate now in a more rigorous manner that the instability is suppressed for any reflectivity function, provided that the following inequality is obeyed

$$\mu \leq m \qquad (15)$$

In particular, this will be the case as soon as the cutoff fulfills condition (3).

The demonstration relies upon the dispersion relation (9) which reads for the impedance function, for $\operatorname{Im}\omega > 0$

$$Z[\omega] = \omega^2 \int \frac{\mathrm{d}\omega'}{\pi} \frac{Z_R[\omega']}{\omega'^{\,2}(\epsilon - i\omega + i\omega')} + \frac{ik}{\omega} - im\omega$$

Using Laplace transforms instead of Fourier transforms

$$f\{p\} = f[ip] \qquad \operatorname{Re}p > 0$$

this relation can be transformed to

$$Z\{p\} = \int \mathrm{d}\Phi(\rho)\frac{1 - ip\rho}{p - i\rho} + \frac{k}{p} + p(m - \mu) \qquad (16)$$

with μ defined by equations (12) and (11) and the spectral function Φ given by

$$\mathrm{d}\Phi(\rho) = \frac{Z_R[\rho]\mathrm{d}\rho}{\pi(1 + \rho^2)}$$

One knows that Z_R is positive at all frequencies (see eqs 8 and 14), that the function Φ is bounded (this follows from the finiteness of μ) and that the coefficients k and $m - \mu$ are positive (when $\mu \leq m$). Then, equation (16) is actually the spectral representation for a passive function, that is a function obeying the property [6]

$$\operatorname{Re}Z\{p\} \geq 0 \qquad \text{for } \operatorname{Re}p > 0 \qquad (17)$$

Causality and stability follow from passivity [10], so that the problems associated with runaway solutions are eliminated.

It has to be emphasized that the passivity condition (17) is more stringent than the already known positivity property (8): as an example, the expressions corresponding to the perfect mirror obey condition (8) but not condition (17).

It is also worth to stress that the passivity property is obtained only when the mass m of the mirror is greater than the induced mass μ which is similar to an electromagnetic mass. As far as electron theory is concerned, the same conclusion is reached by Dekker [11]. In other words, passivity is obeyed by the impedance Z but not by the contribution of the motional susceptibility taken separately [4], in contrast with the usual models of Brownian motion where the motional contribution itself is a passive function [6,12].

ENERGY CONSIDERATIONS

Passivity is equivalent to the proposition that a positive amount of energy is transfered from the reservoir (source of the applied force F_a) to the moving mirror [6]

$$W_a(t) = \int_{-\infty}^{t} \mathrm{d}t' \; F_a(t')v(t') \geq 0 \qquad (18)$$

One immediately deduces from the equation of motion (2) that

$$W_a(t) = \Delta E(t) + W_m(t)$$
$$\Delta E(t) = E(t) - E(-\infty)$$
$$E(t) = \frac{1}{2}kq(t)^2 + \frac{1}{2}mv(t)^2$$
$$W_m(t) = -\int_{-\infty}^{t} \mathrm{d}t' \; F_m(t')v(t')$$

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[4]At low frequencies $\frac{\chi[\omega]}{i\omega} \approx m\tau\omega^2\Gamma[0]$, so that this function does not obey the passivity property (17) in the vicinity of $\omega = 0$.

The energy W_a provided by the reservoir is the sum of an energy ΔE stored in the mirror's motion and of a part W_m radiated in the vacuum state.

When we consider a situation where the force F_a is applied only during a limited time interval [2] and where the initial and final states of the mirror are the same ($\Delta E(\infty) = 0$), we conclude from condition (18) that the net radiated energy is positive

$$W_a(\infty) = W_m(\infty) \geq 0$$

Thus, passivity appears to be connected to the fact that vacuum can damp the mirror's motion but cannot excite it. This is the ultimate reason why stability follows from passivity: the vacuum cannot provide energy for sustaining runaway solutions.

CONCLUSION

The expression (1) of the damping force for a perfect mirror in vacuum is associated with unacceptable runaway solutions. We have shown that this problem does not take place for a realistic mirror with a reflectivity cutoff obeying condition (3).

The impedance function can be computed explicitly, neglecting the recoil effect, and is found to be a passive function, so that stability is ensured. This is connected to the fact that no energy can be extracted from the vacuum state.

It is tempting to consider that passivity is actually a general property of the impedance function for a scatterer in vacuum. Then, a consistent treatment of a perfect mirror, taking into account the recoil effect for reflected fields, would necessarily provide us with a passive function. This would also be true for a satisfactory treatment of the electron. Attempts in this direction, unsuccessful up to now, are discussed by Rueda [5].

[1] Jaekel M.T. and Reynaud S., *Quant. Opt.* **4** 39 (1992).
[2] Fulling S.A. and Davies P.C.W., *Proc. R. Soc. London* **A348** 393 (1976).
[3] Unruh W.G. in *Quantum Optics, Experimental Gravitation and Measurement Theory*, eds P. Meystre and M.O. Scully (Plenum, 1983), p.647; Jaekel M.T. and Reynaud S., *EuroPhys. Lett.* **13** 301 (1990); Braginski V.B. and Khalili F.Ya, *Phys. Lett.* **A161** 197 (1991).
[4] Rohrlich F., *Classical Charged Particles* (Addison Wesley, 1965).
[5] Rueda A., *Phys. Rev.* **A30** 2221 (1984).
[6] Meixner J., in *Statistical Mechanics of Equilibrium and Non Equilibrium*, ed. J. Meixner (North Holland, 1965), p.52.
[7] Jaekel M.T. and Reynaud S., *J. Physique* **I 1** 1395 (1991).
[8] Jaekel M.T. and Reynaud S., *J. Physique* **I 2** 149 (1992).
[9] Landau L.D. and Lifshitz E.M., *Course of Theoretical Physics: Statistical Physics* (Pergamon, 1958) ch.12.
[10] Nussenzveig H.M., *Causality and dispersion relations* (Academic Press, 1972).
[11] Dekker H., *Phys. Lett.* **A107** 255 (1985).
[12] Ford G.W., Kac M. and Mazur P., *J. Math. Phys.* **6** 504 (1965); Haken H., *Rev. Mod. Phys.* **47** 67 (1975); Dekker H., *Phys. Rep.* **80** 1 (1981); Caldeira A.O. and Leggett A.J., *Ann. Phys.* **149** 374 (1983); Ford G.W., Lewis J.T. and O'Connell R.F., *Phys. Rev. Lett.* **55** 2273 (1985).